关于E(A,B)为Abel群的证明*

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【摘要】本文给出了E(A,B)的定义，利用了加法范畴，Abel范畴中关于积、余积、拉回、推出的定义以及结论，给出了相应的乘法，并且证明了，在这个乘法下，E(A,B)构成群。

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Abstract

In this paper, E (A, B) is defined using the additive category, Abel category on the plot, coproduct, pullback, the introduction of definitions and conclusions, gives the corresponding multiplication, and proved in this Multiply next, E (A, B) constitute a group.
keywords:product coproduct pullbact pushout

目录

引言

在同调代数中，Ext 函子是Hom 函子的导函子，$Ext^n : \mathcal{C} \to Abel$ 群范畴。此函子首见于代数拓扑，但其应用遍布许多领域。我们一般都是对$\mathcal{C}$是具有足够多的内射对象的$Abel$范畴中才能通过内射分解得到Ext^n函子，比如环R的左模范畴$R-Mod$中，但具有足够多的内射对象的$Abel$范畴不是一般存在的，这样就对Ext^n函子无法在一般的$Abel$范畴中使用。我们知道在环R的左模范畴$R-Mod$中，$Ext^1(A,B)$可与建立群结构的A通过B的扩张集合$E(A,B)$有群同构，对一般$Abel$范畴中的扩张集合中的群结构$E(A,B)$，基本上大家都认为它是一个群，但基本上很难在文献中找到证明。本文就这个问题给出一个证明，从而对一般Ext函子的建立打下基础。

第1章　定义

Able范畴$\mathcal{C}$中，A,B$\in$ ob($\mathcal{C}$)令

$$T(A,B) = \{0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0\text{正合} | C \in ob(\mathcal{C})\}$$

对任意$\xi, \eta \in T(A,B)$,$\xi : 0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0, \eta : 0 \longrightarrow B \longrightarrow D \longrightarrow A \longrightarrow 0$，定义相等关系$\sim$

$$\xi \sim \eta \ \Leftrightarrow g \in \mathcal{C}(C,D)\text{使得下图可交换：}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow g & & \| & & \\
0 & \longrightarrow & B & \longrightarrow & D & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

容易验证关系$\sim$是等价关系，令E$(A,B) = {}^{T(A,B)}\!/\!_{\sim}$,记$\xi \in T(A,B)$的等价类为$[\xi]$。$\xi, \eta \in T(A,B), \xi : 0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0, \eta : 0 \longrightarrow B \longrightarrow D \longrightarrow A \longrightarrow 0$,令$q_a = q_1 + q_2 \in \mathcal{C}(A, A \bigoplus A)$，其中$q_1, q_2$为余射影。令$p_b = p_1' + p_2' \in \mathcal{C}(B \bigoplus B, B)$，其中$p_1', p_2'$是射影。可以得到交换图：

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & Y & \longrightarrow & A & \longrightarrow & 0 \\
 & & \uparrow p_b & ② & \uparrow & & \| & & \\
0 & \longrightarrow & B \bigoplus B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & ① & \downarrow q_a & & \\
0 & \longrightarrow & B \bigoplus B & \longrightarrow & C \bigoplus D & \longrightarrow & A \bigoplus A & \longrightarrow & 0
\end{array}$$

其中①和②分别为拉回和推出，而每一行都为正合列（从后面的命题14、15、16得到），$0 \longrightarrow B \longrightarrow Y \longrightarrow A \longrightarrow 0$记为$\xi * \eta$，从而$\xi * \eta \in T(A,B)$，定义$[\xi] + [\eta] = [\xi * \eta]$。

E(A,B)在运算$+$下为Abel群，需验证

(1) 运算合理性：当$[\xi_1] = [\xi_2], [\eta_1] = [\eta_2]$时，有$[\xi_1 * \eta_1] = [\xi_2 * \eta_2]$；

(2) 存在单位元：存在$[e] \in E(A,B)$使得任意$[\xi] \in E(A,B), [\xi] + [e] = [e] + [\xi] = [\xi]$；

(3) 存在逆元：对任意$[\xi] \in E(A,B)$存在$[\xi] \in E(A,B)$，使得$[\xi]+[\eta] = [\eta]+[\xi] = [e]$；

(4) 结合律：对任意$[\xi], [\eta], [\zeta] \in E(A,B)$，有$([\xi] + [\eta]) + [\zeta] = [\xi] + ([\eta] + [\zeta])$；

(5) 交换律：对任意$[\xi], [\eta] \in E(A,B), [\xi] + [\eta] = [\eta] + [\xi]$。

第2章　常用命题

命题 1 设$\mathcal{C}$是一个准加法范畴，$A,B,C\in ob\mathcal{C}$下列条件等价：

(1) C是A与B的积；

(2) C是A与B的余积；

(3) 存在态射$p_1:C\longrightarrow A,p_2:C\longrightarrow B,q_1:A\longrightarrow C,q_2:B\longrightarrow C$ 满足

$$p_1q_1=1_A,p_2q_2=1_B;q_1p_1+q_2p_2=1_C$$

证明　见文献[1]。 ■

附注 1 p_1,p_2实际上为射影，q_1,q_2为余射影。从积、余积的万有性质以及上述命题可知，对任意$f_1:A_1\to B_1,f_2:A_2\to B_2$，存在唯一$f_1\bigoplus f_2:A_1\bigoplus A_2\to B_1\bigoplus B_2$使得交换图成立：$i=1,2$

$$\begin{tikzcd}
A_1\bigoplus A_2 \arrow[r, "f_1\bigoplus f_2"] \arrow[d, "p_i"'] & B_1\bigoplus B_2 \arrow[d, "p_i'"] & A_1\bigoplus A_2 \arrow[r, "f_1\bigoplus f_2"] & B_1\bigoplus B_2 \\
A_i \arrow[r, "f_i"'] & B_i & A_i \arrow[u, "q_i"] \arrow[r, "f_i"'] & B_i \arrow[u, "q_i'"']
\end{tikzcd}$$

且容易知道$f_1\bigoplus f_2$为单（满）射$\Leftrightarrow f_1,f_2$为单（满）射。命题对多个对象时也成立。

命题 2 设$\mathcal{C}$是一个准加法范畴，$f_i:A_i\longrightarrow B_i,g_i:B_i\longrightarrow C_i(i=1,2)$则有：

(1) $(g_1\bigoplus g_2)(f_1\bigoplus f_2)=(g_1f_1)\bigoplus(g_2f_2)$

(2) $1_{A_1}\bigoplus 1_{A_2}=1_{A_1\bigoplus A_2}$

证明　先证(1)，由命题1知有以下交换图：

$$\begin{tikzcd}
A_1\bigoplus A_2 \arrow[r, "f_1\bigoplus f_2"] \arrow[d, "p_{ai}"'] & B_1\bigoplus B_2 \arrow[r, "g_1\bigoplus g_2"] \arrow[d, "p_{bi}"] & C_1\bigoplus C_2 \arrow[d, "p_{ci}"] \\
A_i \arrow[r, "f_i"'] & B_i \arrow[r, "g_i"'] & C_i
\end{tikzcd}$$

$$\begin{tikzcd}
A_1\bigoplus A_2 \arrow[r, "g_1f_1\bigoplus g_2f_2"] \arrow[d, "p_{ai}"'] & C_1\bigoplus C_2 \arrow[d, "p_{ci}"] \\
A_i \arrow[r, "g_if_i"'] & C_i
\end{tikzcd}$$

则对$i=1,2$有：

$$p_{ci}((f_1\bigoplus f_2)(g_1\bigoplus g_2))=g_ip_{bi}(f_1\bigoplus f_2)=g_if_ip_{ai}$$

$$p_{ci}(g_1f_1\bigoplus g_2f_2)=g_if_ip_{ai}$$

从而由射影的集体单性质可得：$(g_1 \bigoplus g_2)(f_1 \bigoplus f_2) = (g_1 f_1) \bigoplus (g_2 f_2)$。关于(2)，有交换图：

$$\begin{array}{ccc} A_1 \bigoplus A_2 & \xrightarrow{id_{A_1} \bigoplus id_{A_2}} & A_1 \bigoplus A_2 \\ {\scriptstyle p_{ai}}\downarrow & & \downarrow{\scriptstyle p_{ai}} \\ A_i & \xrightarrow[id_{A_i}]{} & A_i \end{array}$$

对$i = 1, 2$有

$$p_{ai}(id_{A_1} \bigoplus id_{A_2}) = id_{A_i} p_{ai} = p_{ai}$$

$$p_{ai}(id_{A_1 \bigoplus A_2}) = p_{ai}$$

从而由射影的集体单性质可得$id_{A_1} \bigoplus id_{A_2} = id_{A_1 \bigoplus A_2}$。 ■

推论 1 准加法范畴$\mathcal{C}$中，有交换图：$i = 1, 2$

$$\begin{array}{ccc} X & \xrightarrow{f_i} & C \\ {\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle k_i} \\ B & \xrightarrow[g_i]{} & A \end{array}$$

则可得到交换图：

$$\begin{array}{ccc} A_1 \bigoplus A_2 & \xrightarrow{f_1 \bigoplus f_2} & B_1 \bigoplus B_2 \\ {\scriptstyle h_1 \bigoplus h_2}\downarrow & & \downarrow{\scriptstyle k_1 \bigoplus k_2} \\ C_1 \bigoplus C_2 & \xrightarrow[g_1 \bigoplus g_2]{} & D_1 \bigoplus D_2 \end{array}$$

推论 2 准加法范畴$\mathcal{C}$中，$f_i : A_i \to B_i, (i = 1, 2)$为同构，则$f_1 \bigoplus f_2$为同构，且$(f_1 \bigoplus f_2)^{-1} = f_1^{-1} \bigoplus f_2^{-1}$

命题 3 准加法范畴$\mathcal{C}$中，有拉回方形$i = 1, 2$

$$\begin{array}{ccc} X_i & \xrightarrow{\bar{f}_i} & C_i \\ {\scriptstyle \bar{g}_i}\downarrow & & \downarrow{\scriptstyle g_i} \\ B_i & \xrightarrow[f_i]{} & A_i \end{array}$$

则有拉回方形

$$\begin{array}{ccc} X_1 \bigoplus X_2 & \xrightarrow{\bar{f}_1 \bigoplus \bar{f}_2} & C_1 \bigoplus C_2 \\ {\scriptstyle \bar{g}_1 \bigoplus \bar{g}_2}\downarrow & & \downarrow{\scriptstyle g_1 \bigoplus g_2} \\ B_1 \bigoplus B_2 & \xrightarrow[f_1 \bigoplus f_2]{} & A_1 \bigoplus A_2 \end{array}$$

证明 交换性从推论1可得，主要证万有性质。
若$s:Y\to B_1\bigoplus B_2$，及$t:Y\to C_1\bigoplus C_2$有$(f_1\bigoplus f_2)s=(g_1\bigoplus g_2)t$，下证存在唯一态射$h:Y\to X_1\bigoplus X_2$，使得下列交换图成立：$(i=1,2)$

$$\begin{tikzcd}
Y \arrow[dr, dotted, "h"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & & \\
& X_1\bigoplus X_2 \arrow[r, "\bar{f}_1\bigoplus\bar{f}_2"] \arrow[d, "\bar{g}_1\bigoplus\bar{g}_2"] & C_1\bigoplus C_2 \arrow[r, "p_{ci}"] \arrow[d, "g_1\bigoplus g_2"] & C_i \arrow[dd, "g_i"] \\
& B_1\bigoplus B_2 \arrow[r, "f_1\bigoplus f_2"] \arrow[d, "p_{bi}"] & A_1\bigoplus A_2 \arrow[dr, "p_{ai}"] & \\
& B_i \arrow[rr, "f_i"'] & & A_i
\end{tikzcd}$$

由$(f_1\bigoplus f_2)s=(g_1\bigoplus g_2)t$，则$p_{ai}(f_1\bigoplus f_2)s=p_{ai}(g_1\bigoplus g_2)t$，即有$f_i(p_{bi}s)=g_i(p_{ci}t)$，而由拉回方形的万有性质，则存在唯一的态射$h_i:Y\to X_i$使得有下面交换图成立：

$$\begin{tikzcd}
Y \arrow[dr, dotted, "h_i"] \arrow[drr, bend left, "p_{ci}t"] \arrow[ddr, bend right, "p_{bi}s"'] & & \\
& X_i \arrow[r, "\bar{f}_i"] \arrow[d, "\bar{g}_i"] & C_i \arrow[d, "g_i"] \\
& B_i \arrow[r, "f_i"'] & A_i
\end{tikzcd}$$

即有$p_{bi}s=\bar{g}_ih_i$和$p_{ci}t=\bar{f}_ih_i$成立。而从$X_1\bigoplus X_2$的万有性质，则存在唯一的态射$h:Y\to X_1\bigoplus X_2$使得有下面交换图：

$$\begin{tikzcd}
& X_1\bigoplus X_2 \arrow[dl, "p_{x1}"'] \arrow[dr, "p_{x2}"] & \\
X_1 & & X_2 \\
& Y \arrow[ul, "h_1"] \arrow[ur, "h_2"'] \arrow[uu, dotted, "h"] &
\end{tikzcd}$$

下证所得h满足条件，即有$s=(\bar{g}_1\bigoplus\bar{g}_2)h$和$t=(\bar{f}_1\bigoplus\bar{f}_2)h$。从交换图：

$$\begin{tikzcd}
X_1\bigoplus X_2 \arrow[r, "\bar{g}_1\bigoplus\bar{g}_2"] \arrow[d, "p_{xi}"'] & B_1\bigoplus B_2 \arrow[d, "p_{bi}"] & X_1\bigoplus X_2 \arrow[r, "\bar{f}_1\bigoplus\bar{f}_2"] \arrow[d, "p_{xi}"'] & C_1\bigoplus C_2 \arrow[d, "p_{ci}"] \\
X_i \arrow[r, "\bar{g}_i"'] & B_i & X_i \arrow[r, "\bar{f}_i"'] & C_i
\end{tikzcd}$$

有$(i=1,2)$

$$p_{bi}=\bar{g}_ih_i=\bar{g}_ip_{xi}h=p_{bi}(\bar{g}_1\bigoplus\bar{g}_2)h$$

$$p_{ci}=\bar{f}_ih_i=\bar{f}_ip_{xi}h=p_{ci}(\bar{f}_1\bigoplus\bar{f}_2)h$$

从而由射影的集体单性质可得

$$s = (\bar{g}_1 \bigoplus \bar{g}_2)h$$

$$t = (\bar{f}_1 \bigoplus \bar{f}_2)h$$

从而命题得证。 ■

对偶地，有

命题 4 准加法范畴$\mathcal{C}$中，有推出方形$i = 1, 2$

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle \bar{g}_i} \\ C_i & \xrightarrow[\bar{f}_i]{} & Y_i \end{array}$$

则有推出方形

$$\begin{array}{ccc} A_1 \bigoplus A_2 & \xrightarrow{f_1 \oplus f_2} & B_1 \bigoplus B_2 \\ {\scriptstyle g_1 \oplus g_2}\downarrow & & \downarrow{\scriptstyle \bar{g}_1 \oplus \bar{g}_2} \\ C_1 \bigoplus C_2 & \xrightarrow[\bar{f}_1 \oplus \bar{f}_2]{} & Y_1 \bigoplus Y_2 \end{array}$$

■

特别的，当有其中一个拉回方形是平凡时，即对任意的态射$h : D \to E$有拉回方形，同时也是推出方形

$$\begin{array}{ccc} D & \xrightarrow{h} & E \\ {\scriptstyle id_D}\downarrow & & \downarrow{\scriptstyle id_E} \\ D & \xrightarrow[h]{} & E \end{array}$$

有相应结论。

命题 5 准加法范畴$\mathcal{C}$中，交换图：

$$\begin{array}{ccc} X_2 & \xrightarrow{\bar{\bar{f}}} & D \\ {\scriptstyle \bar{h}}\downarrow & & \downarrow{\scriptstyle h} \\ X_1 & \xrightarrow{\bar{f}} & C \\ {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & A \end{array}$$

其中两个小方形为拉回方形，则大方形：

$$\begin{array}{ccc} X_2 & \xrightarrow{\bar{\bar{f}}} & D \\ {\scriptstyle \bar{g}\bar{h}}\downarrow & & \downarrow{\scriptstyle gh} \\ B & \xrightarrow[f]{} & A \end{array}$$

为拉回方形。

证明 首先由拉回方形的交换性

$$h\bar{\bar{f}} = \bar{f}\bar{h}$$

$$g\bar{f} = f\bar{g}$$

可得$f\bar{g}\bar{h} = g\bar{f}\bar{h} = gh\bar{\bar{f}}$，即大方形的交换性成立，下面证明万有性质。
若有态射$s: Y \to B, t: Y \to D$使得：$fs = ght$，下证$\exists 1\ H: Y \to X_2$使得下面交换图成立：

$$\begin{tikzcd}
Y \arrow[dr, dotted, "H"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\
& X_2 \arrow[r, "\bar{\bar{f}}"] \arrow[d, "\bar{g}\bar{h}"] & X \arrow[d, "gh"] \\
& B \arrow[r, "f"'] & A
\end{tikzcd}$$

而由下面小方形为拉回方形可得存在唯一的态射$G: Y \to X_1$使得：$s = \bar{g}G, ht = \bar{f}G$，再用上面小方形为拉回方形，存在唯一的态射$H: Y \to X_2$使得$G = \bar{h}H, t = \bar{\bar{f}}H$，从而$s = \bar{g}\bar{h}H, t = \bar{\bar{f}}H$，从而命题成立。

$$\begin{tikzcd}
Y \arrow[dr, dotted, "H"] \arrow[ddr, dotted, "G"] \arrow[drr, bend left, "t"] \arrow[dddr, bend right, "s"'] & & \\
& X_2 \arrow[r, "\bar{\bar{f}}"] \arrow[d, "\bar{h}"] & D \arrow[d, "h"] \\
& X_1 \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"] & C \arrow[d, "g"] \\
& B \arrow[r, "f"] & A
\end{tikzcd}$$

■

对偶地，有

命题 6 准加法范畴$\mathcal{C}$中，交换图：

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & D \arrow[d, "\bar{g}"] \\
C \arrow[r, "\bar{f}"] \arrow[d, "h"'] & Y_1 \arrow[d, "\bar{h}"] \\
D \arrow[r, "\bar{\bar{f}}"'] & Y_2
\end{tikzcd}$$

其中两个小方形为推出方形，则大方形：

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "hg"'] & D \arrow[d, "\bar{h}\bar{g}"] \\
B \arrow[r, "\bar{\bar{f}}"'] & Y_2
\end{tikzcd}
$$

为推出方形。 ■

为了方便叙述，我们称态射$s_1 : (A_1 \bigoplus A_2) \bigoplus A_3 \to A_1 \bigoplus A_2 \bigoplus A_3$是自然的，是指有下面交换图成立：

$$
\begin{tikzcd}
 & & & (A_1 \bigoplus A_2) \bigoplus A_3 \arrow[dll, "p_1''"'] \arrow[ddr, "p_2''"] \arrow[ddd, dotted, "s_1"] & \\
 & A_1 \bigoplus A_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & & \\
A_1 & & A_2 & & A_3 \\
 & & & A_1 \bigoplus A_2 \bigoplus A_3 \arrow[ulll, "p_1'"] \arrow[ul, "p_2'"'] \arrow[ur, "p_3'"'] &
\end{tikzcd}
$$

$$
\begin{cases}
p_1 p_1'' = p_1' s_1 \\
p_2 p_1'' = p_2' s_1 \\
p_2'' = p_3' s_1
\end{cases}
$$

和

$$
\begin{tikzcd}
 & & & (A_1 \bigoplus A_2) \bigoplus A_3 \arrow[ddd, dotted, "s_1"] & \\
 & A_1 \bigoplus A_2 \arrow[urr, "q_1''"] & & & \\
A_1 \arrow[ur, "q_1"] \arrow[drrr, "q_1'"'] & & A_2 \arrow[ul, "q_2"] \arrow[dr, "q_2'"] & & A_3 \arrow[uul, "q_2''"'] \arrow[dl, "q_3'"] \\
 & & & A_1 \bigoplus A_2 \bigoplus A_3 &
\end{tikzcd}
$$

$$
\begin{cases}
q_1' = s_1 q_1'' q_1 \\
q_2' = s_1 q_1'' q_2 \\
q_3' = s_1 q_2''
\end{cases}
$$

同样地，可定义态射$s_2 : A_1 \bigoplus (A_2 \bigoplus A_3) \to A_1 \bigoplus A_2 \bigoplus A_3$的自然性。

命题 7 加法范畴$\mathcal{C}$中，对任意$A_1, A_2, A_3 \in ob(\mathcal{C})$，存在唯一的态射$s_1 : (A_1 \bigoplus A_2) \bigoplus A_3 \to A_1 \bigoplus A_2 \bigoplus A_3$同构，且自然。

证明 由积的定义，可以得到下面交换图：

$$
\begin{tikzcd}
 & & & (A_1\bigoplus A_2)\bigoplus A_3 \arrow[dll, "p_1''"'] \arrow[ddr, "p_2''"] \arrow[ddd, dotted, bend left=10, "s_1"] & \\
 & A_1\bigoplus A_2 \arrow[dl, "p_1"'] \arrow[dr, bend left, "p_2"] & & & \\
A_1 & & A_2 & & A_3 \\
 & & & A_1\bigoplus A_2\bigoplus A_3 \arrow[ulll, "p_1'"] \arrow[ul, bend left, "p_2'"] \arrow[ur, "p_3'"'] \arrow[uull, dotted, bend right, "h"] \arrow[uuu, dotted, bend left=10, "t_1"] &
\end{tikzcd}
$$

由于$A_1\bigoplus A_2$为积，则存在唯一的态射$h:A_1\bigoplus A_2\bigoplus A_3\to A_1\bigoplus A_2$使得$p_1'=p_1h,p_2'=p_2h$。存在唯一的态射$t_1:A_1\bigoplus A_2\bigoplus A_3\to(A_1\bigoplus A_2)\bigoplus A_3$使得：$h=p_1''t_1,p_3'=p_2''t_1$。从而有

$$
\begin{cases}
p_1'=p_1p_1''t_1\\
p_2'=p_2p_1''t_1\\
p_3'=p_2''t_1
\end{cases}
$$

而由$A_1\bigoplus A_2\bigoplus A_3$为积，则存在唯一的态射 $s_1:(A_1\bigoplus A_2)\bigoplus A_3\to A_1\bigoplus A_2\bigoplus A_3$使得

$$
\begin{cases}
p_1p_1''=p_1's_1\\
p_2p_1''=p_2's_1\\
p_2''=p_3's_1
\end{cases}
$$

从而可得$p_i'=p_i's_1t_1\ (i=1,2,3)$，而射影为集体单可得$s_1t_1=id_{A_1\bigoplus A_2\bigoplus A_3}$。
再由余积的定义，可以得到下面交换图：

$$
\begin{tikzcd}
 & & & (A_1\bigoplus A_2)\bigoplus A_3 \arrow[ddd, dotted, bend left=10, "s_1'"] & \\
 & A_1\bigoplus A_2 \arrow[urr, "q_1''"] \arrow[ddrr, dotted, bend left, "h'"] & & & \\
A_1 \arrow[ur, "q_1"] \arrow[drrr, "q_1'"'] & & A_2 \arrow[ul, bend right, "q_2"'] \arrow[dr, bend left, "q_2'"] & & A_3 \arrow[uul, "q_2''"'] \arrow[dl, "q_3'"] \\
 & & & A_1\bigoplus A_2\bigoplus A_3 \arrow[uuu, dotted, bend left=10, "t_1'"] &
\end{tikzcd}
$$

由$A_1\bigoplus A_2$为余积，则存在唯一的态射 $h':A_1\bigoplus A_2\to A_1\bigoplus A_2\bigoplus A_3$使得$q_1'=h'q_1,q_2'=hq_2$。由$(A_1\bigoplus A_2)\bigoplus A_3$为余积，则存在唯一的态射 $s_1':(A_1\bigoplus A_2)\bigoplus A_3\to A_1\bigoplus A_2\bigoplus A_3$使得$h'=s_1q_1'',q_3=s_1q_2''$从而有

$$
\begin{cases}
q_1'=s_1'q_1''q_1\\
q_2'=s_1'q_1''q_2\\
q_3'=s_1'q_2''
\end{cases}
$$

由$A_1\bigoplus A_2\bigoplus A_3$为余积，则存在唯一的态射$t_1':A_1\bigoplus A_2\bigoplus A_3\to(A_1\bigoplus A_2)\bigoplus A_3$使得

$$
\begin{cases}
q_1''q_1=t_1'q_1'\\
q_1''q_2=t_1'q_2'\\
q_2''=t_1'q_3'
\end{cases}
$$

从而有

$$
\begin{aligned}
s_1 &= \sum_{i=1}^{3} q_i' p_i' s_1 \\
&= (s_1' q_1'' q_1)(p_1 p_1'') + (s_1' q_1'' q_2)(p_2 p_1'') + (s_1' q_2'') p_2'' \\
&= s_1' [q_1''(q_1 p_1 + q_2 p_2) p_1'' + q_2'' p_2''] \\
&= s_1'
\end{aligned}
$$

同理，有

$$
\begin{aligned}
t_1' &= t_1' \sum_{i=1}^{3} q_i' p_i' \\
&= (q_1'' q_1)(p_1 p_1'' t_1) + (q_1'' q_2)(p_2 p_1'' t_1) + q_2''(p_2'' t_1) \\
&= [q_1''(q_1 p_1 + q_2 p_2) p_1'' + q_2'' p_2''] t_1 \\
&= t_1
\end{aligned}
$$

则可得

$$
\begin{aligned}
t_1' s_1 &= t_1' (\sum_{i=1}^{3} q_i' p_i') s_1 \\
&= (q_1'' q_1)(p_1 p_1'') + (q_1'' q_2)(p_2 p_1'') + q_2'' p_2'' \\
&= id_{(A_1 \bigoplus A_2) \bigoplus A_3}
\end{aligned}
$$

即$t_1 s_1 = id_{(A_1 \bigoplus A_2) \bigoplus A_3}$，从而$s_1$为同构。 ■

同样地，有

命题 8 加法范畴$\mathcal{C}$中，对任意$A_1, A_2, A_3 \in ob(\mathcal{C})$，存在唯一的态射$s_2 : A_1 \bigoplus (A_2 \bigoplus A_3) \to A_1 \bigoplus A_2 \bigoplus A_3$同构，且自然。

命题 9 加法范畴$\mathcal{C}$中，$f_i : A_i \to B_i$,则有交换图：

$$
\begin{tikzcd}
(A_1 \bigoplus A_2) \bigoplus A_3 \arrow[r, "(f_1 \bigoplus f_2) \bigoplus f_3"] \arrow[d, "s_1"'] & (B_1 \bigoplus B_2) \bigoplus B_3 \arrow[d, "s_1'"] \\
A_1 \bigoplus A_2 \bigoplus A_3 \arrow[r, "f_1 \bigoplus f_2 \bigoplus f_3"'] & B_1 \bigoplus B_2 \bigoplus B_3
\end{tikzcd}
$$

其中 $\begin{array}{l} s_1 : (A_1 \bigoplus A_2) \bigoplus A_3 \to A_1 \bigoplus A_2 \bigoplus A_3 \\ s_1' : (B_1 \bigoplus B_2) \bigoplus B_3 \to B_1 \bigoplus B_2 \bigoplus B_3 \end{array}$ 。

证明 $i = 1, 2$时，可以得到下面交换图：

$$
\begin{tikzcd}
& A_1 \bigoplus A_2 \arrow[rr, "f_1 \bigoplus f_2"] \arrow[dd, dotted, "p_{ai}"] & & B_1 \bigoplus B_2 \arrow[dd, "p_{bi}"] \\
(A_1 \bigoplus A_2) \bigoplus A_3 \arrow[ur, "p_{a1}''"] \arrow[rr, "(f_1 \bigoplus f_2) \bigoplus f_3"] \arrow[dd, "s_1"'] & & (B_1 \bigoplus B_2) \bigoplus B_3 \arrow[ur, "p_{b1}''"] \arrow[dd, "s_1'"] & \\
& A_i \arrow[rr, dotted, "f_i"] & & B_i \\
A_1 \bigoplus A_2 \bigoplus A_3 \arrow[ur, dotted, "p_{ai}'"] \arrow[rr, "f_1 \bigoplus f_2 \bigoplus f_3"'] & & B_1 \bigoplus B_2 \bigoplus B_3 \arrow[ur, dotted, "p_{bi}'"'] &
\end{tikzcd}
$$

则有

$$\begin{aligned}
p'_{bi}(f_1 \bigoplus f_2 \bigoplus f_3)s_1 &= f_i p'_{ai} s_1 \\
&= f_i p_{ai} p''_{a1} \\
&= p_{bi}(f_1 \bigoplus f_2) p''_{a1} \\
&= p_{bi} p''_{b1}[(f_1 \bigoplus f_2) \bigoplus f_3] \\
&= p'_{bi} s'_1[(f_1 \bigoplus f_2) \bigoplus f_3]
\end{aligned}$$

$i = 3$时，可以得到下面交换图：

$$\begin{tikzcd}
 & (A_1 \bigoplus A_2) \bigoplus A_3 \arrow[rr, "(f_1 \oplus f_2) \oplus f_3"] \arrow[ddl, "s_1"'] \arrow[dr, dotted, "p''_{a2}"] & & (B_1 \bigoplus B_2) \bigoplus B_3 \arrow[dr, "p''_{b2}"] \arrow[ddl, "s'_1"'] & \\
 & & A_3 \arrow[rr, dotted, "f_3"] & & B_3 \\
A_1 \bigoplus A_2 \bigoplus A_3 \arrow[urr, dotted, "p'_{a3}"] \arrow[rr, "f_1 \oplus f_2 \oplus f_3"'] & & B_1 \bigoplus B_2 \bigoplus B_3 \arrow[urr, "p'_{b3}"'] & &
\end{tikzcd}$$

$$\begin{aligned}
p'_{b3}(f_1 \bigoplus f_2 \bigoplus f_3)s_1 &= f_3 p'_{a3} s_1 \\
&= f_3 p''_{a2} \\
&= p''_{b2}[(f_1 \bigoplus f_2) \bigoplus f_3] \\
&= p'_{bi} s'_1[(f_1 \bigoplus f_2) \bigoplus f_3]
\end{aligned}$$

从而由射影的集体单性质可得：$(f_1 \oplus f_2 \oplus f_3)s_1 = s'_1[(f_1 \oplus f_2) \oplus f_3]$，即命题成立。 ■

类似地可以得到

命题 10 加法范畴$\mathcal{C}$中，$f_i : A_i \to B_i$,则有交换图：

$$\begin{tikzcd}
A_1 \bigoplus (A_2 \bigoplus A_3) \arrow[r, "f_1 \oplus (f_2 \oplus f_3)"] \arrow[d, "s_2"'] & B_1 \bigoplus (B_2 \bigoplus B_3) \arrow[d, "s'_2"] \\
A_1 \bigoplus A_2 \bigoplus A_3 \arrow[r, "f_1 \oplus f_2 \oplus f_3"'] & B_1 \bigoplus B_2 \bigoplus B_3
\end{tikzcd}$$

其中 $\begin{matrix} s_2 : A_1 \bigoplus (A_2 \bigoplus A_3) \to A_1 \bigoplus A_2 \bigoplus A_3 \\ s'_2 : B_1 \bigoplus (B_2 \bigoplus B_3) \to B_1 \bigoplus B_2 \bigoplus B_3 \end{matrix}$ 为自然同构。

命题 11 *(五项引理)*
*Abel*范畴$\mathcal{C}$中的图表：

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "h_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\
B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & B_3 \arrow[r, "g_3"] & B_4 \arrow[r, "g_4"] & B_5
\end{tikzcd}$$

若该图表是交换的并且上下两个横行都是正合的，则

(1) 如果h_2, h_4是单态射，h_1是满态射，则h_3是单态射。

(2) 如果h_2, h_4是满态射，h_5是单态射，则h_3是满态射。

(3) 如果h_1, h_2, h_3, h_4, h_5均是同构，则h_3是同构。

证明 见文献[1] ■

命题 12 *Abel*范畴$\mathcal{C}$中的一个拉回（推出）方形：

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

若*k*是满（单）态射，则*f*是满（单）态射，且该方形是一个推出（拉回）方形。

证明 见文献[1] ■

由于单态射族保持拉回，满态射族保持推出（基本性质），从而可知，Able范畴中拉回和推出保持态射的单性和满性。

命题 13 *Abel*范畴$\mathcal{C}$中，有正合列：

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

和拉回方形：

$$\begin{array}{ccc} B' & \xrightarrow{\overline{g}} & C' \\ {\scriptstyle \overline{h}}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow[g]{} & C \end{array}$$

则存在唯一的态射$k : A \to B'$*s.t.*下图为正合列同态：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{k} & B' & \xrightarrow{\overline{g}} & C' & \longrightarrow & 0 \\ & & \| & & {\scriptstyle \overline{h}}\downarrow & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

证明 见文献[2] ■

对偶地，有

命题 14 *Abel*范畴$\mathcal{C}$中，有正合列：

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

和推出方形：

$$\begin{array}{ccc} A' & \xrightarrow{\overline{f}} & B' \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle \overline{h}} \\ A & \xrightarrow[f]{} & B \end{array}$$

则存在唯一的态射$k : B' \to C'$使得下图为正合列同态：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\bar{f}} & B' & \xrightarrow{k} & C' & \longrightarrow & 0 \\ & & h\Big\uparrow & & \bar{h}\Big\uparrow & & \Big\| & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

命题 15 *Abel*范畴$\mathcal{C}$中，有正合列

$$0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0(i = 1, 2)$$

则有正合列

$$0 \longrightarrow A_1 \bigoplus A_2 \xrightarrow{f_1 \bigoplus f_2} B_1 \bigoplus B_2 \xrightarrow{g_1 \bigoplus g_2} C_1 \bigoplus C_2 \longrightarrow 0$$

证明 由f_1, f_2为单态射，g_1, g_2为满态射，可得：$f_1 \bigoplus f_2$为单态射，$g_1 \bigoplus g_2$为满态射，从而只需证$f_1 \bigoplus f_2 = ker(g_1 \bigoplus g_2)$即可。看下图：

$$\begin{tikzcd} & P \arrow[dl, dotted, "\bar{h}"'] \arrow[d, "h"] \arrow[ddl, dotted, "h_i"] & & \\ 0 \arrow[r] & A_1 \bigoplus A_2 \arrow[r, "f_1 \bigoplus f_2"] \arrow[d, "p_{ai}"'] & B_1 \bigoplus B_2 \arrow[r, "g_1 \bigoplus g_2"] \arrow[d, "p_{bi}"'] & C_1 \bigoplus C_2 \arrow[r] \arrow[d, "p_{ci}"'] & 0 \\ 0 \arrow[r] & A_i \arrow[r, "f_i"] & B_i \arrow[r, "g_i"] & C_i \arrow[r] & 0 \end{tikzcd}$$

首先证$(g_1 \bigoplus g_2)(f_1 \bigoplus f_2) = 0$。当i=1,2时，有

$$\begin{aligned} p_{ci}(g_1 \bigoplus g_2)(f_1 \bigoplus f_2) &= g_i p_{bi}(f_1 \bigoplus f_2) \\ &= (g_i f_i)p_{ai} \\ &= 0 \end{aligned}$$

则$(g_1 \bigoplus g_2)(f_1 \bigoplus f_2) = 0$。
下面证万有性质，设$h : P \to B_1 \bigoplus B_2$使得：$(g_1 \bigoplus g_2)h = 0$，则有$p_{ci}(g_1 \bigoplus g_2)h = g_i(p_{bi}h) = 0$，且由$f_i = ker(g_i)$可得存在唯一的态射$h_i : P \to A_i$使得$f_i h_i = p_{bi}h$。而由于$A_1 \bigoplus A_2$为积，则存在唯一的态射$\bar{h} : P \to A_1 \bigoplus A_2$使得$h_i = p_i\bar{h}$。从而对$i = 1, 2$有

$$p_{bi}(f_1 \bigoplus f_2)\bar{h} = f_i p_{ai}\bar{h} = f_i h_i = p_{bi}h$$

则可得$(f_1 \bigoplus f_2)\bar{h} = h$，从而$f_1 \bigoplus f_2 = ker(g_1 \bigoplus g_2)$。
综上所述，命题成立。 ■

推论 3 *Abel*范畴$\mathcal{C}$中，有正合列：$(i = 1, 2, 3)$

$$0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$$

则

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (A_1 \bigoplus A_2) \bigoplus A_3 & \xrightarrow{(f_1 \bigoplus f_2) \bigoplus f_3} & (B_1 \bigoplus B_2) \bigoplus B_3 & \xrightarrow{(g_1 \bigoplus g_2) \bigoplus g_3} & (C_1 \bigoplus C_2) \bigoplus C_3 & \longrightarrow & 0 \\ & & \Big\downarrow s_1 & & \Big\downarrow s_1' & & \Big\downarrow s_1'' & & \\ 0 & \longrightarrow & A_1 \bigoplus A_2 \bigoplus A_3 & \xrightarrow{f_1 \bigoplus f_2 \bigoplus f_3} & B_1 \bigoplus B_2 \bigoplus B_3 & \xrightarrow{g_1 \bigoplus g_2 \bigoplus g_3} & C_1 \bigoplus C_2 \bigoplus C_3 & \longrightarrow & 0 \end{array}$$

和

$$\begin{tikzcd}
0 \arrow[r] & A_1\bigoplus(A_2\bigoplus A_3) \arrow[r, "f_1\bigoplus(f_2\bigoplus f_3)"] \arrow[d, "s_2"] & B_1\bigoplus(B_2\bigoplus B_3) \arrow[r, "g_1\bigoplus(g_2\bigoplus g_3)"] \arrow[d, "s_2'"] & C_1\bigoplus(C_2\bigoplus C_3) \arrow[r] \arrow[d, "s_2''"] & 0 \\
0 \arrow[r] & A_1\bigoplus A_2\bigoplus A_3 \arrow[r, "f_1\bigoplus f_2\bigoplus f_3"] & B_1\bigoplus B_2\bigoplus B_3 \arrow[r, "g_1\bigoplus g_2\bigoplus g_3"] & C_1\bigoplus C_2\bigoplus C_3 \arrow[r] & 0
\end{tikzcd}$$

都为同构，其中$s_1, s_1', s_1'', s_2, s_2', s_2''$都为自然同构。

第3章　证明

3.1　运算的良定义与交换性

引理 1 *Abel*范畴$\mathcal{C}$中，若有正合列同构：

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "T_1"'] & B_1 \arrow[r] \arrow[d, "T_2"] & C\bigoplus C \arrow[r] \arrow[d, "s"] & 0 \\
0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C\widetilde{\bigoplus} C \arrow[r] & 0
\end{tikzcd}$$

其中$C\bigoplus C$ 和$C\widetilde{\bigoplus}C$都是C,C的直和，且$\tilde{q}_c = sq_c$，则由$q_c, \tilde{q}_c$拉回产生的正合列，存在唯一的态射$G: X_1 \to X_2$ 使得有正合列同构：

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "T_1"'] & X_1 \arrow[r] \arrow[d, "G"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_2 \arrow[r] & X_2 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

证明　看下图：

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "h_1"] \arrow[d, equal] & X_1 \arrow[r, "\bar{g}_1"] \arrow[d, "\bar{q}_c"] \arrow[ddd, dotted, bend left, "G"] & C \arrow[r] \arrow[d, "q_c"] \arrow[ddd, equal, bend left] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "T_1"'] & B_1 \arrow[r, "g_1"] \arrow[d, "T_2"] & C\bigoplus C \arrow[r] \arrow[d, "s"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "f_2"] \arrow[d, equal] & B_2 \arrow[r, "g_2"] & C\bigoplus C \arrow[r] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "h_2"] & X_2 \arrow[r, "\bar{g}_2"] \arrow[u, "\bar{\tilde{q}}_c"] & C \arrow[r] \arrow[u, "\tilde{q}_c"] & 0
\end{tikzcd}$$

取id_c，则$sq_c = \tilde{q}_c id_c = \tilde{q}_c$。

对态射$T_2\bar{q}_c : X \to B_2$，由于$(X_2, \bar{g}_2, \bar{\tilde{q}}_c)$为拉回，$g_2T_2\bar{q}_c = g_1\bar{q}_c = q_c\bar{g}_1$，则存在唯一的态射$G: X_1 \to X_2$使得

$$T_2\bar{q}_c = \bar{\tilde{q}}_c G$$

$$\bar{g}_1 = \bar{g}_2 G$$

且有

$$\bar{\tilde{q}}_c h_2 T_1 = f_2 T_1 = T_2 f_1 = T_2 \bar{\tilde{q}}_c \bar{h}_1 = \bar{\tilde{q}}_c G h_1$$

而$\bar{\tilde{q}}_c$为单态射，则有$h_2T_1 = Gh_1$。从五项引理，且T, id_c为同构，可知G为同构。从而命题得证。 ■

对偶地，有

引理 2 *Abel*范畴$\mathcal{C}$中，若有正合列同构：

$$\begin{tikzcd}
0 \arrow[r] & A\bigoplus A \arrow[r] \arrow[d, "s'"'] & B_1 \arrow[r] \arrow[d, "T_1"] & C_1 \arrow[r] \arrow[d, "T_2"] & 0 \\
0 \arrow[r] & A\widetilde{\bigoplus} A \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & 0
\end{tikzcd}$$

其中$A\bigoplus A$ 和$A\widetilde{\bigoplus} A$都是*A,A*的直和，且$p_a = \tilde{p}_a s'$，则由$p_a, \tilde{p}_a$推出产生的正合列，存在唯一的态射$H : Y_1 \to Y_2$ 使得有正合列同构：

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & Y_1 \arrow[r] \arrow[d, "H"] & C_1 \arrow[r] \arrow[d, "T_2"] & 0 \\
0 \arrow[r] & A \arrow[r] & Y_2 \arrow[r] & C_2 \arrow[r] & 0
\end{tikzcd}$$

附注 2 在引理*1*中，$q_c = q_1 + q_2, \tilde{q}_c = \tilde{q}_1 + \tilde{q}_2$，而$q_1, q_2$为*C*到直和$C\bigoplus C$的余射影，$\tilde{q}_1, \tilde{q}_2$为*C*到$C\widetilde{\bigoplus} C$的余射影；而在引理*2*中，$p_a = p_1 + p_2, \tilde{p}_a = \tilde{p}_1 + \tilde{p}_2$，而$p_1, p_2$为直和$A\bigoplus A$到*C*的余射影，$\tilde{p}_1, \tilde{p}_2$为$A\widetilde{\bigoplus} A$到*A*的射影。

定理 1 *Abel*范畴$\mathcal{C}$中，$\xi_1, \xi_2, \eta_1, \eta_2 \in T(A, B)$，其中$\xi_1 \sim \xi_2, \eta_1 \sim \eta_2, i = 1, 2$

$$\xi_i : 0 \longrightarrow B \longrightarrow C_i \longrightarrow A \longrightarrow 0$$

$$\eta_i : 0 \longrightarrow B \longrightarrow D_i \longrightarrow A \longrightarrow 0$$

则$\xi_1 * \eta_1 \sim \xi_2 * \eta_2$。

证明 Abel范畴中，对正合列：$i = 1, 2$

$$0 \longrightarrow B \xrightarrow{f_i} C_i \xrightarrow{g_i} A \longrightarrow 0$$

可取到s, s', s'' 使得有下面的正合列同构：（证明是自然的）

$$\begin{tikzcd}
0 \arrow[r] & B\bigoplus B \arrow[r] \arrow[d, "s"'] & C_1\bigoplus C_2 \arrow[r] \arrow[d, "s'"] & A\bigoplus A \arrow[r] \arrow[d, "s''"] & 0 \\
0 \arrow[r] & B\widetilde{\bigoplus} B \arrow[r] & C_1\widetilde{\bigoplus} C_2 \arrow[r] & A\widetilde{\bigoplus} A \arrow[r] & 0
\end{tikzcd}$$

且s, s''保持引理1,2的条件，从而先用$q_a, \tilde{q}_a$拉回得到同构：

$$\begin{tikzcd}
0 \arrow[r] & B\bigoplus B \arrow[r] \arrow[d, "s"'] & X_1 \arrow[r] \arrow[d, "G"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B\widetilde{\bigoplus} B \arrow[r] & X_2 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

再用$p_b, \widetilde{p}_b$拉回得到同构：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & Y_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle H} & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & Y_2 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

从而保证在T(A,B)中，做“$*$”时，不管取哪个直和，得到的结果在E(A,B)是一致的。那么在E(A,B)中的两个元素$[\xi],[\eta]$，对ξ,η进行$*$时，我们可以使相同对象的直和固定，不影响在E(A,B)中的结果。

对$[\xi_1]=[\xi_2],[\eta_1]=[\eta_2]\in E(A,B)$，其中：$i=1,2$

$$\xi_i : 0 \longrightarrow B \longrightarrow C_i \longrightarrow A \longrightarrow 0$$

$$\eta_i : 0 \longrightarrow B \longrightarrow D_i \longrightarrow A \longrightarrow 0$$

则有同构：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B\bigoplus B & \longrightarrow & C_1\bigoplus D_1 & \longrightarrow & A\bigoplus A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & B\bigoplus B & \longrightarrow & C_2\bigoplus D_2 & \longrightarrow & A\bigoplus A & \longrightarrow & 0 \end{array}$$

则先用q_a拉回，再用p_b推出，由引理1,2可得到同构：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & Y_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle H} & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & Y_2 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

■

由此可知在$E(A,B)$定义的运算是良定义的。

定理 2 （交换性）*Abel*范畴$\mathcal{C}$中，$[\xi],[\eta]\in E(A,B)$，则$[\xi]+[\eta]=[\eta]+[\xi]$。

证明 对$[\xi],[\eta]\in E(A,B)$，设：

$$\xi : 0 \longrightarrow B \xrightarrow{f_1} C \xrightarrow{g_1} A \longrightarrow 0$$

$$\eta : 0 \longrightarrow B \xrightarrow{f_2} D \xrightarrow{g_2} A \longrightarrow 0$$

则存在s_a, s_b, s使得有同构：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B\bigoplus B & \xrightarrow{f_1\bigoplus f_2} & C\bigoplus D & \xrightarrow{g_1\bigoplus g_2} & A\bigoplus A & \longrightarrow & 0 \\ & & {\scriptstyle s_b}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s_a} & & \\ 0 & \longrightarrow & B\bigoplus B & \xrightarrow{f_2\bigoplus f_1} & D\bigoplus C & \xrightarrow{g_2\bigoplus g_1} & A\bigoplus A & \longrightarrow & 0 \end{array}$$

且$\widetilde{q}_a = s_a q_a, p_b = \widetilde{p}_b s_b$

$$\begin{tikzcd} & A\bigoplus A \arrow[dd, dotted, "s_a"] & \\ A \arrow[ur, "q_{a1}"] \arrow[dr, "\widetilde{q}_{a1}"'] & & A \arrow[ul, "q_{a2}"'] \arrow[dl, "\widetilde{q}_{a2}"] \\ & A\bigoplus A & \end{tikzcd} \qquad \begin{tikzcd} & B\bigoplus B \arrow[dl, "p_{b1}"'] \arrow[dr, "p_{b2}"] \arrow[dd, dotted, "s_b"] & \\ B & & B \\ & B\bigoplus B \arrow[ul, "\widetilde{p}_{b1}"] \arrow[ur, "\widetilde{p}_{b2}"'] & \end{tikzcd}$$

则从而利用引理1,2可得到$[\eta * \xi] = [\xi * \eta]$，即$[\eta] + [\xi] = [\xi] + [\eta]$。从而交换性得证。■

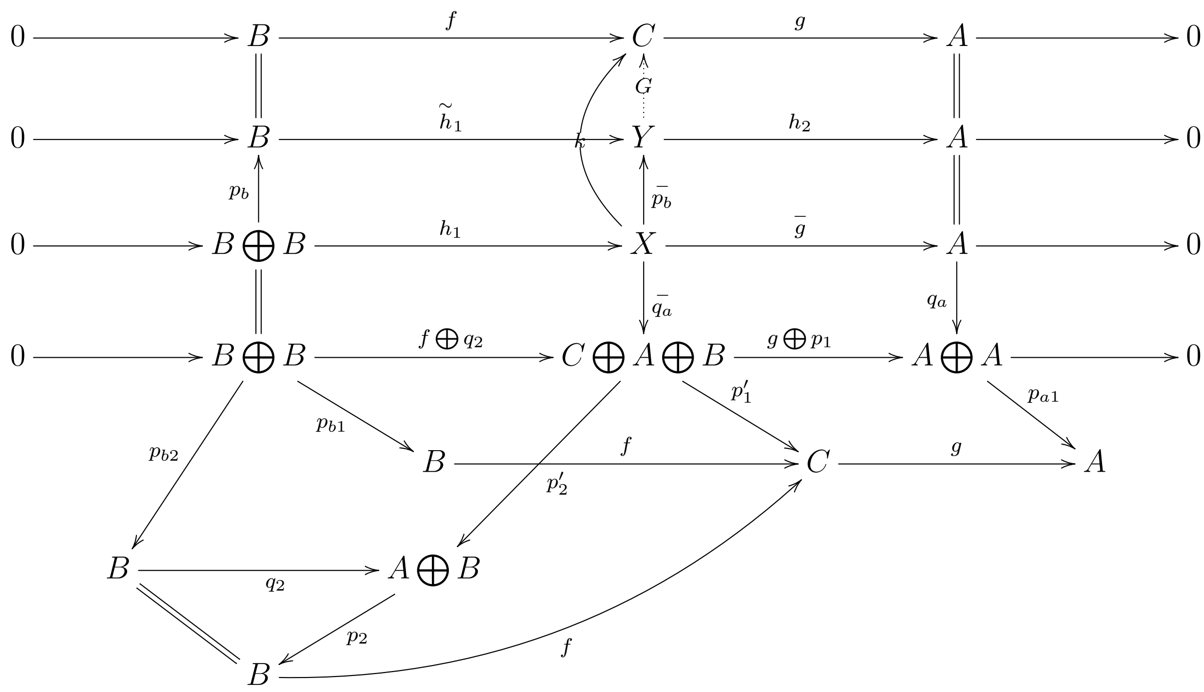
3.2 单位元

定理 3 *Abel*范畴$\mathcal{C}$中，$e : 0 \longrightarrow B \longrightarrow A \bigoplus B \longrightarrow A \longrightarrow 0$正合，且对任意$[\eta] \in E(A, B)$, 有$[e] + [\eta] = [\eta] + [e] = [\eta]$,即$[e]$是单位元。

证明 由交换性质，只需证明：$[\eta] + [e] = [\eta]$即可。令

$$\eta : 0 \longrightarrow B \xrightarrow{f} C \xrightarrow{g} A \longrightarrow 0$$

看下图，其中$(X, \bar{g}, \bar{q_a})$为拉回，$(Y, \tilde{h_1}, \bar{p_b})$为推出：



从而可知

$$\eta * e : 0 \longrightarrow B \xrightarrow{\tilde{h_1}} Y \xrightarrow{h_2} A \longrightarrow 0$$

令$k = p_1'\bar{q_a} + fp_2p_2'\bar{q_a}$，则有

$$\begin{aligned} kh_1 &= p_1'\bar{q_a}h_1 + fp_2p_2'\bar{q_a}h_1 \\ &= p_1'(f \bigoplus q_2) + fp_2p_2'(f \bigoplus \bar{q}') \\ &= fp_{b1} + fp_2q_2p_{b2} \\ &= fp_{b1} + fp_{b2} \\ &= fp_b \end{aligned}$$

由$(Y, \tilde{h_1}, \bar{p_b})$为推出，则存在唯一的态射$G : Y \to C$使得

$$f = G\tilde{h_1}$$

$$k = p_1'\bar{q_a} + fp_2p_2'\bar{q_a} = G\bar{p_b}$$

下面证$gG = h_2$。从

$$\begin{aligned}
gG\overline{p_b} &= gk \\
&= g(p_1'\overline{q_a} + fp_2p_2'\overline{q_a}) \\
&= gp_1'\overline{q_a} + (gf)p_2p_2'\overline{q_a} \\
&= gp_1'\overline{q_a} \\
&= p_{a1}(g \bigoplus p_1)\overline{q_a} \\
&= p_{a1}q_a\overline{g} \\
&= id_A\overline{g} \\
&= \overline{g} \\
&= h_2\overline{p_b}
\end{aligned}$$

且p_b为满态射可知$\overline{p_b}$也为满态射，从而$gG = h_2$，由此可知(id_b, G, id_a)为正合列的同态，也为同构，从而$[\eta * e] = [\eta]$，即$[\eta] + [e] = [\eta]$，命题成立。 ■

3.3 逆元

定理 4 *Abel*范畴$\mathcal{C}$中，ξ, η正合列，且

$$\xi : 0 \longrightarrow B \xrightarrow{f} C \xrightarrow{g} A \longrightarrow 0$$

$$\eta : 0 \longrightarrow B \xrightarrow{f} C \xrightarrow{-g} A \longrightarrow 0$$

则$[\xi] + [\eta] = [\eta] + [\xi] = [e]$，从而可知$[\eta] \in E(A, B)$为元素$[\xi]$的逆元。

证明

$$\xi : 0 \longrightarrow B \xrightarrow{f} C \xrightarrow{g} A \longrightarrow 0$$

$$\eta : 0 \longrightarrow B \xrightarrow{f} C \xrightarrow{-g} A \longrightarrow 0$$

由$ker(g) = ker(-g)$可知：若ξ正合，则η正合。看下图：

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "q_2"] & A \bigoplus B \arrow[r, "p_1"] & A \arrow[r] & 0 \\
0 \arrow[r] & B \bigoplus B \arrow[u, "p_b"] \arrow[r, "\varphi"] \arrow[d, equal] & C \bigoplus B \arrow[u, "g \bigoplus id_b"'] \arrow[r, "gp_2'"] \arrow[d, "t"] & A \arrow[u, equal] \arrow[r] \arrow[d, "q_a"] & 0 \\
0 \arrow[r] & B \bigoplus B \arrow[r, "f \bigoplus f"] & C \bigoplus C \arrow[r, "g \bigoplus (-g)"] & A \bigoplus A \arrow[r] & 0
\end{tikzcd}$$

其中$p_2' : C \bigoplus B \to B$为射影。

1.下面先证存在$t : C \bigoplus B \to C \bigoplus C$使得$(C \bigoplus B, gp_2', t)$为拉回。看下图：

$$\begin{tikzcd}
 & C \bigoplus B \arrow[dd, dotted, "s"] & \\
C \arrow[ur, "q_1'"] \arrow[dr, "-id_c"'] & & B \arrow[ul, "q_2'"'] \arrow[dl, "f"] \\
 & C &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C \bigoplus C \arrow[dl, "p_{c1}"'] \arrow[dr, "p_{c2}"] & \\
C & & C \\
 & C \bigoplus B \arrow[uu, dotted, "t"] \arrow[ul, "p_1'"] \arrow[ur, "s"'] &
\end{tikzcd}$$

由于$C\bigoplus B, C\bigoplus C$既是积也是余积，从而存在唯一的态射$s: C\bigoplus B\to C$使得$-id_c = sq'_1, f = sq'_2$，也存在唯一的态射$t: C\bigoplus B\to C\bigoplus C$使得$p'_1 = p_{c1}t, s = p_{c2}t$。

对交换性，看下图：

$$\begin{tikzcd}
& C\bigoplus B \arrow[rr, "gp'_1"] \arrow[d, "t"'] & & A \arrow[d, "q_a"] & \\
& C\bigoplus C \arrow[rr, "g\bigoplus(-g)"] \arrow[dr, "p_{c1}"] \arrow[ddl, "p_{c2}"'] & & A\bigoplus A \arrow[dr, "p_{a1}"] \arrow[ddl, "p_{a2}"] & \\
& & C \arrow[rr, "g"'] & & A \\
C \arrow[rr, "-g"'] & & A & &
\end{tikzcd}$$

而有

$$p_{a1}(g\bigoplus(-g))t = gp_{b1}t = gp'_1$$

$$p_{a1}q_a gp'_1 = p_{a1}q_{a1}gp'_1 = gp'_1$$

$$p_{a1}(g\bigoplus(-g))t = gp_{b1}t = gp'_1$$

$$p_{a1}q_a gp'_1 = p_{a2}q_{a2}gp'_1 = gp'_1$$

$$-gs = gs(q'_1p'_1 + q'_2p'_2) = -g(-id_c p'_1 + fp'_2) = gp'_1$$

从而可得$(g\bigoplus(-g))t = q_a gp'_1$，即交换性成立。

对万有性质，设$r_1: X\to C\bigoplus C, r_2: X\to A$使得$(g\bigoplus(-g))r_1 = q_a r_2$，看下图：

$$\begin{tikzcd}
X \arrow[drrrr, bend left, "r_2"] \arrow[drr, dotted, "h"] \arrow[ddrr, bend right, "r_1"'] \arrow[dddd, dotted, "k"] & & & & & \\
& & C\bigoplus B \arrow[rr, "gp'_1"] \arrow[d, "t"'] & & A \arrow[d, "q_a"] & \\
& & C\bigoplus C \arrow[rr, "g\bigoplus(-g)"] \arrow[dr, "p_{c1}"] \arrow[ddl, "p_c"'] & & A\bigoplus A \arrow[dr, "p_{a1}"] \arrow[ddl] & \\
& & & C \arrow[rr, "g", "p_a^- = p_{a1} - p_{a2}"'] & & A \\
B \arrow[r, "f"] & C \arrow[rr, "g"] & & A & &
\end{tikzcd}$$

先证$gp_c = p_a^-(g\bigoplus(-g)), p_a^- q_a = 0$，

$$\begin{aligned}
p_a^-(g\bigoplus(-g)) &= p_{a1}(g\bigoplus(-g)) - p_{a2}(g\bigoplus(-g)) \\
&= gp_{c1} - (-g)p_{c2} \\
&= g(p_{c1} + p_{c2}) \\
&= gp_c
\end{aligned}
\qquad
\begin{aligned}
p_a^- q \quad & (p_{a1} - p_{a2})(q_{a1} + q_{a2}) \\
&= p_{a1}q_{a1} - pa2qa2 \\
&= id_a - id_a \\
&= 0
\end{aligned}$$

则$gp_cr_1 = p_a^-(g\bigoplus(-g)), p_a^-r_1 = p_a^-q_ar_2 = 0$。而$f = ker(g)$，则存在唯一的态射$k : X \to B$使得$p_cr_1 = fk$。
从$C\bigoplus B$为积，则存在唯一的态射$h : X \to C\bigoplus B$使得下面交换图成立：

$$\begin{tikzcd}
 & C\bigoplus B \arrow[dl, "p_1'"'] \arrow[dr, "p_2'"] & \\
C & & B \\
 & X \arrow[ul, "p_{c1}r_1"] \arrow[ur, "k"'] \arrow[uu, dotted, "h"] &
\end{tikzcd}$$

从

$$p_{c1}th = p_1'h = p_{c1}r_1$$

$$\begin{aligned}
p_{c2}th &= sh \\
&= s(q_1'p_1' + q_2'p_2')h \\
&= -id_cp_1'h + fp_2'h \\
&= -p_{c1}r_1 + fk \\
&= -p_{c1}r_1 + p_cr_1 \\
&= p_{c2}r_1
\end{aligned}$$

可知$th = r_1$，而

$$\begin{aligned}
r_2 &= p_{a1}q_ar_2 \\
&= p_{a1}(g\bigoplus(-g))r_1 \\
&= gp_{c1}r_1 \\
&= gp_1'h
\end{aligned}$$

即得到$r_1 = th, r_2 = gp_1'h$。

综上所述，$(C\bigoplus B, gp_2', t)$为拉回。

2.下面证存在$\varphi : B\bigoplus B \to C\bigoplus B$使得$(id_{B\bigoplus B}, t, q_a)$为正合列的同态，且使得$(A\bigoplus B, q_2, g\bigoplus id_b)$为推出。看下图：

$$\begin{tikzcd}
 & C\bigoplus B \arrow[dl, "p_1'"'] \arrow[dr, "p_2'"] & \\
C & & B \\
 & B \arrow[ul, "f"] \arrow[ur, "id_b"'] \arrow[uu, dotted, "\psi"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B\bigoplus B \arrow[dd, dotted, "\varphi"] & \\
B \arrow[ur, "q_{b1}"] \arrow[dr, "\psi"'] & & B \arrow[ul, "q_{b2}"'] \arrow[dl, "q_2'"] \\
 & C\bigoplus B &
\end{tikzcd}$$

由于$C\bigoplus B, B\bigoplus B$既是积也是余积，从而存在唯一的态射$\psi : B \to C\bigoplus B$使得$f = p_1'\psi, id_b = p_2'\psi$，也存在唯一的态射$\varphi : B\bigoplus B \to C\bigoplus B$使得$\psi = \varphi q_{b1}, q_2' = \varphi p_{b2}$。

(a)先证$(id_{B\bigoplus B}, t, q_a)$是正合列的同态，由于$t$为单态射，且$(C\bigoplus B, gp_2', t)$为拉回，所以只需证明$t\varphi = f\bigoplus f$即可。看下图：

$$\begin{tikzcd}
B \arrow[r, "q_{b1}"'] \arrow[d, "f"] \arrow[dr, "\psi"] \arrow[dd, bend right=60, "f"'] & B\bigoplus B \arrow[l, bend right, "p_{b1}"'] \arrow[r, bend left, "p_{b2}"] \arrow[d, "\varphi"] \arrow[dd, bend right=40, "f\bigoplus f"'] & B \arrow[l, "q_{b2}"] \arrow[dl, "q_2'"'] \arrow[d, equal] \arrow[dd, bend left=60, "f"] \\
C \arrow[d, equal] & C\bigoplus B \arrow[l, "p_1'"'] \arrow[r, "p_2'"] \arrow[d, "t"] \arrow[dl, "p_1'"] \arrow[dr, "s"] & B \arrow[d, "f"'] \\
C & C\bigoplus C \arrow[l, "p_{c1}"] \arrow[r, "p_{c2}"'] & C
\end{tikzcd}$$

有

$$p_1'\varphi q_{b1} = p_1'\psi = f = fp_{b1}q_{b1}$$
$$p_1'\varphi q_{b2} = p_1'q_2' = 0 = fp_{b1}q_{b2}$$
$$s\varphi q_{b1} = s\psi = s(q_1'p_1' + q_2'p_2')\psi = -id_c f + f id_b = 0 = fp_{b2}q_{b1}$$
$$s\varphi q_{b2} = sq_2' = f = f id_b = fp_{b1}q_{b2}$$

有$p_1'\varphi = fp_{b2}, s\varphi = fp_{b2}$，则有

$$p_{c1}t\varphi = p_1'\varphi = fp_{b1} = p_{c1}(f \bigoplus f)$$

$$p_{c2}t\varphi = s\varphi = fp_{b2} = p_{c2}(f \bigoplus f)$$

从而$t\varphi = f \bigoplus f$，$(id_{B\bigoplus B}, t, q_a)$是正合列的同态。

(b)下面证$(A \bigoplus B, q_2, g \bigoplus id_b)$为推出。先证交换性，即$(g \bigoplus id_b)\varphi = q_2p_b$。看下图：

$$\begin{tikzcd}
A & A \bigoplus B \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \\
C \arrow[u, "g"] & C \bigoplus B \arrow[l, "p_1'"'] \arrow[r, "p_2'"] \arrow[u, "g \bigoplus id_b"'] & B \arrow[u, equal] \arrow[l, bend right=0, phantom] \\
B \arrow[u, "f"] \arrow[ur, "\psi"] \arrow[r, "q_{b1}"] & B \bigoplus B \arrow[u, "\varphi"] \arrow[l, bend left, "p_{b1}"] \arrow[r, bend right, "p_{b2}"'] & B \arrow[l, "q_{b2}"'] \arrow[u, equal] \arrow[ul, "q_2'"']
\end{tikzcd}$$

$$p_2'\varphi q_{b1} = p_2'\psi = id_b = p_{b1}q_{b1} = p_bq_{b1}$$
$$p_2'\varphi q_{b2} = p_2'q_2' = id_b = p_{b2}q_{b2} = p_bq_{b2}$$

有$p_2'\varphi = p_b$，则有

$$p_1(g \bigoplus id_b)\varphi = gp_1'\varphi = gfp_{b1} = 0 = p_1q_2p_b$$

$$p_2(g \bigoplus id_b)\varphi = p_2'\varphi = p_b = p_2q_2p_b$$

从而有$(g \bigoplus id_b)\varphi = q_2p_b$，即交换性成立。再证万有性质，设$r_1 : B \to Y, r2 : C \bigoplus B \to Y$使得$p_br_1 = r_2\varphi$，看下图：

$$\begin{tikzcd}
 & & & Y & & \\
0 \arrow[r] & B \arrow[r, "q_2"] \arrow[urr, bend left, "r_1"] & A \bigoplus B \arrow[r, "p_1"] \arrow[ur, dotted, "h"] & A \arrow[r] & 0 \\
0 \arrow[r] & B \bigoplus B \arrow[r, "\varphi"] \arrow[u, "p_b"] & C \bigoplus B \arrow[r, "gp_1'"] \arrow[u, "g \bigoplus id_b"'] \arrow[uur, bend right, "r_2"'] & A \arrow[r] \arrow[u, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "f"] \arrow[u, "q_b^- = q_{b1} - q_{b2}"] & C \arrow[r, "g"] \arrow[u, "q_1'"'] & A \arrow[r] \arrow[u, equal] \arrow[uuu, dotted, bend right, "k"'] & 0
\end{tikzcd}$$

从ψ, φ满足的等式，容易得到

$$p_1'\varphi q_b^- = p_2'(q_{b1} - q_{b2}) = p_2'(\psi - q_2') = f = p_1'q_1'f$$

$$p_2'\varphi q_b^- = p_2'(q_{b1} - q_{b2}) = p_2'(\psi - q_2') = id_b - id_b = 0 = p_2'q_1'f$$

则有$\varphi q_b^- = q_1'f$，而且$gp_1'q_1' = g$，从而(q_b^-, q_1', id_a)是正合列的同态。另一方面，由于$coker(f) = g$，且有

$$\begin{aligned} r_2q_1'f &= r_2\varphi q_b^- \\ &= r_1p_bq_b^- \\ &= r_1(p_{b1} + pb2)(q_{b1} - q_{b2}) \\ &= r_1(id_b - id_b) \\ &= 0 \end{aligned}$$

则存在唯一的态射$k : A \to Y$使得$r_2q_1' = kg$。由于$A \bigoplus B$为余积，则存在唯一的态射$h : A \bigoplus B \to Y$使得有交换图：

$$\begin{tikzcd} & A \bigoplus B \arrow[dd, dotted, "h"] & \\ A \arrow[ur, "q_1"] \arrow[dr, "k"'] & & B \arrow[ul, "q_2"'] \arrow[dl, "r_1"] \\ & Y & \end{tikzcd}$$

$$k = hq_1$$

$$r_1 = hq_2$$

从而只需证明$r_2 = h(g \bigoplus id_b)$即可得到$(A \bigoplus B, q_2, g \bigoplus id_b)$为推出。由于有下图成立：

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "q_1'"'] & A \arrow[d, "q_1"] \\ C \bigoplus B \arrow[r, "g \bigoplus id_b"] & A \bigoplus B \\ B \arrow[u, "q_2'"] \arrow[r, "id_b"] & B \arrow[u, "q_2"'] \end{tikzcd}$$

则有

$$\begin{cases} h(g \bigoplus id_b)q_1' = hq_1g = kg = r_2q_1' \\ h(g \bigoplus id_b)q_2' = hq_2 = r1 = r_1p_bq_{b2} = r_2\varphi q_b = r_2q_2' \end{cases}$$

从而，$r_2 = h(g \bigoplus id_b)$，$(A \bigoplus B, q_2, g \bigoplus id_b)$为推出。

从而可得$[e] = [\xi * \eta]$，即$[\xi] + [\eta] = [e]$，且由交换性可知命题成立。 ∎

3.4 结合律

引理 3 *Abel*范畴$\mathcal{C}$中，$\xi : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$，$h : D \to C$为单态射，$k : A \to E$为满态射，则$\xi$先用$h$拉回再用$k$推出可产生正合列：$\eta_1 : 0 \longrightarrow E \longrightarrow Y \longrightarrow D \longrightarrow 0$，而$\xi$ 先用k推出再用k拉回可产生正合列$\eta_2 : 0 \longrightarrow E \longrightarrow X \longrightarrow D \longrightarrow 0$，且存在$G : Y \to X$使得有同构：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & Y & \longrightarrow & D & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle G} & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & X & \longrightarrow & D & \longrightarrow & 0 \end{array}$$

证明 看下图：

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, "\bar{t}"] \arrow[dddd, equal, bend right] & Y_1 \arrow[r, "v"] \arrow[dddd, dotted, bend left, "G"] & D \arrow[r] \arrow[d, equal] \arrow[dddd, equal, bend left] & 0 \\
0 \arrow[r] & A \arrow[r, "t"] \arrow[u, "k"] \arrow[d, equal] & X_1 \arrow[r, "\bar{g}"] \arrow[u, "\bar{k}"] \arrow[d, "\bar{h}"] \arrow[ddd, dotted, bend right, "H"'] & D \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "k"] & B \arrow[r, "g"] \arrow[d, "\widetilde{k}"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E \arrow[r, "\widetilde{f}"] \arrow[d, equal] & Y_2 \arrow[r, "s"] & C \arrow[r] & 0 \\
0 \arrow[r] & E \arrow[r, "w"] & X_2 \arrow[r, "\widetilde{s}"] \arrow[u, "\widetilde{h}"] & D \arrow[r] \arrow[u, "h"] & 0
\end{tikzcd}$$

有$\widetilde{k}\bar{h}:X_1\to Y_2,\bar{g}:X_1\to D$，由于$(X_2,\widetilde{h},\widetilde{s})$为拉回，且

$$s\widetilde{k}\bar{h}=g\bar{h}=h\bar{g}$$

则存在唯一的态射$H:X_1\to X_2$使得

$$\widetilde{h}H=\widetilde{k}\bar{h}$$

$$\widetilde{s}H=\bar{g}$$

对$H:X_1\to X_2,w:E\to X_2$，由于$(Y_1,\bar{k},\bar{t})$为推出，且

$$\widetilde{h}Ht=\widetilde{k}\bar{h}t=\widetilde{k}f=\widetilde{f}k=\widetilde{h}wk$$

同时h为单态射可知$\widetilde{h}$为单态射，从而$Ht=wk$，则存在唯一的态射$G:Y_1\to X_2$使得

$$H=G\bar{k}$$

$$w=G\bar{t}$$

下证(id_E,G,id_D)为正合列的同态。由于$w=G\bar{t}$，则只需证$\widetilde{s}G=v$。而

$$\widetilde{s}G\bar{k}=\widetilde{s}H=\bar{g}=v\bar{k}$$

且k为满态射可知$\bar{k}$为满态射，则$\widetilde{s}G=v$，从而(id_E,G,id_D)为同态，且id_E,id_D为同构可知(id_E,G,id_D)为同构。 ■

由此可知，对单态射h和满态射k作用于正合列顺序，得到的正合列为同一类。

引理 4 加法范畴$\mathcal{C}$中，任意$A\in ob(\mathcal{C})$，则有交换图：

$$\begin{tikzcd}
(A\bigoplus A)\bigoplus A \arrow[r, "s_1"] & A\bigoplus A\bigoplus A \\
A \arrow[u, "(q\bigoplus id_A)q"] \arrow[r, equal] & A \arrow[u, "\widetilde{q}=\widetilde{q}_1+\widetilde{q}_2+\widetilde{q}_2"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A\bigoplus A)\bigoplus A \arrow[r, "s_1"] \arrow[d, "p(p\bigoplus id_A)"'] & A\bigoplus A\bigoplus A \arrow[d, "\tilde{p}=\tilde{p}_1+\tilde{p}_2+\tilde{p}_2"] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

证明 注意到对$i=1,2,3$时都有$\tilde{p}_i\tilde{q}=id_A$。看下图：$i=1,2$

$$\begin{tikzcd}
A\bigoplus A \arrow[r, "p_i"] & A \\
(A\bigoplus A)\bigoplus A \arrow[r, "s_1"] \arrow[u, "\tilde{\tilde{p}}_1"] & A\bigoplus A\bigoplus A \arrow[u, "\tilde{p}_i"'] \\
A \arrow[r, equal] \arrow[u, "(q\bigoplus id_A)q"] & A \arrow[u, "\tilde{q}=\tilde{q_1}+\tilde{q_2}+\tilde{q_3}"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "q"] & A\bigoplus A \\
A\bigoplus A \arrow[r, "q\bigoplus id_A"] \arrow[u, "p_1"] \arrow[d, "p_2"'] & (A\bigoplus A)\bigoplus A \arrow[u, "\tilde{\tilde{p}}_1"'] \arrow[d, "\tilde{\tilde{p}}_2"] \\
A \arrow[r, "id_A"'] & A
\end{tikzcd}$$

当$i=1,2$时，有

$$\begin{aligned}
\tilde{p}_i s_1(q\bigoplus id_A)q &= p_i\tilde{\tilde{p}}_1(q\bigoplus id_A)q \\
&= p_iqp_iq \\
&= id_A \\
&= \tilde{p}_i\tilde{q}
\end{aligned}$$

当$i=3$时，有

$$\begin{aligned}
\tilde{p}_3 s_1(q\bigoplus id_A)q &= \tilde{\tilde{p}}_2(q\bigoplus id_A)q \\
&= p_2q \\
&= id_A \\
&= \tilde{p}_3\tilde{q}
\end{aligned}$$

则由射影的集体单性质可知$s_1(q\bigoplus id_A)q=\tilde{q}$，即图交换。另一个方形，看下图：$i=1,2$

$$\begin{tikzcd}
A\bigoplus A \arrow[d, "\tilde{\tilde{q}}_1"'] & A \arrow[l, "q_i"'] \\
(A\bigoplus A)\bigoplus A \arrow[r, "s_1"] & A\bigoplus A\bigoplus A \arrow[u, "\tilde{q}_i"'] \\
A \arrow[r, equal] \arrow[u, "p(p\bigoplus id_A)"] & A \arrow[u, "\tilde{p}=\tilde{p}_1+\tilde{p}_2+\tilde{p}_3"']
\end{tikzcd}$$

$$\begin{tikzcd}
A\bigoplus A \arrow[r, "p"] \arrow[d, "\widetilde{\widetilde{q}}_1"'] & A \arrow[d, "q_1"] \\
(A\bigoplus A)\bigoplus A \arrow[r, "p\bigoplus id_A"] & A\bigoplus A \\
A \arrow[r, "id_A"'] \arrow[u, "\widetilde{\widetilde{q}}_2"] & A \arrow[u, "q_2"']
\end{tikzcd}$$

有

$$p(p\bigoplus id_A)\widetilde{\widetilde{q}}_1 = pq_1p = p$$

当$i=1,2$时，有

$$\begin{aligned}\widetilde{p}s_1\widetilde{\widetilde{q}}_1q_i &= \widetilde{p}\widetilde{q}_i \\ &= id_A \\ &pq_i \\ &p(p\bigoplus id_A)\widetilde{\widetilde{q}}_1q_i\end{aligned}$$

则可得$\widetilde{p}s_1\widetilde{\widetilde{q}}_1 = p(p\bigoplus id_A)\widetilde{\widetilde{q}}_1$。
另一方面，有：

$$p(p\bigoplus id_A)\widetilde{\widetilde{q}}_2 = pq_1 = id_A$$

$$\widetilde{p}s_1\widetilde{\widetilde{q}}_2 = \widetilde{p}\widetilde{q}_3 = id_A$$

从而有$p(p\bigoplus id_A) = \widetilde{p}s_1$，即图交换。综上所述，引理成立。 ■
同样地，有

引理 5 加法范畴$\mathcal{C}$中，$\forall A\in ob(\mathcal{C})$，则有交换图

$$\begin{tikzcd}
A\bigoplus(A\bigoplus A) \arrow[r, "s_2"] & A\bigoplus A\bigoplus A \\
A \arrow[u, "(id_A\bigoplus q)q"] \arrow[r, equal] & A \arrow[u, "\widetilde{q}=\widetilde{q}_1+\widetilde{q}_2+\widetilde{q}_2"']
\end{tikzcd}$$

$$\begin{tikzcd}
A\bigoplus(A\bigoplus A) \arrow[r, "s_2"] \arrow[d, "p(id_A\bigoplus p)"'] & A\bigoplus A\bigoplus A \arrow[d, "\widetilde{p}=\widetilde{p}_1+\widetilde{p}_2+\widetilde{p}_2"] \\
A \arrow[r, equal] & A
\end{tikzcd}$$

引理 6 *Abel*范畴$\mathcal{C}$中，有同构：

$$\begin{tikzcd}
0 \arrow[r] & B_1 \arrow[r] \arrow[d, "T_1"'] & C_1 \arrow[r] \arrow[d, "T_2"] & (A\bigoplus A)\bigoplus A \arrow[r] \arrow[d, "s_1"] & 0 \\
0 \arrow[r] & B \arrow[r] & C_2 \arrow[r] & A\bigoplus A\bigoplus A \arrow[r] & 0
\end{tikzcd}$$

其中$s_1:(A\bigoplus A)\bigoplus A\to A\bigoplus A\bigoplus A$为自然同构。对第一列用$(q_a\bigoplus id_A)q_a:A\to(A\bigoplus A)\bigoplus A$拉回得到正合列：$0\longrightarrow B_1\longrightarrow X_1\longrightarrow A\longrightarrow 0$，第二列用$\widetilde{q}_a:A\to$

$A\bigoplus A\bigoplus A$，则存在态射$G:X_1\to X_2$使得有同构：

$$\begin{tikzcd}
0 \arrow[r] & B_1 \arrow[r] \arrow[d, "T_1"'] & X_1 \arrow[r] \arrow[d, "G"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B_2 \arrow[r] & X_2 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

证明 看下图：

$$\begin{tikzcd}
0 \arrow[r] & B_1 \arrow[r, "h_1"] \arrow[d, equal] & X_1 \arrow[r, "\bar{g}_1"] \arrow[d, "\bar{t}"'] \arrow[ddd, dotted, bend left, "G"] & A \arrow[r] \arrow[d, "(q\bigoplus id_A)q"'] \arrow[ddd, equal, bend left] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "f_1"] \arrow[d, "T_1"'] & C_1 \arrow[r, "g_1"] \arrow[d, "T_2"] & (A\bigoplus A)\bigoplus A \arrow[r] \arrow[d, "s_1"] & 0 \\
0 \arrow[r] & B_2 \arrow[r, "f_2"] \arrow[d, equal] & C_2 \arrow[r, "g_2"] & A\bigoplus A\bigoplus A \arrow[r] & 0 \\
0 \arrow[r] & B_2 \arrow[r, "h_2"] & X_2 \arrow[r, "\bar{g}_2"] \arrow[u, "\bar{\tilde{q}}"] & A \arrow[r] \arrow[u, "\tilde{q}"] & 0
\end{tikzcd}$$

对$\bar{g_1}:X_1\to A, T_2\bar{t}:X_1\to C_2$，由于$(X_2,\bar{g_2},\bar{\tilde{g}})$为拉回方形，且有

$$\tilde{q}\bar{g_1}=s_1(q\bigoplus id_A)q\bar{g_1}=s_1g_1\bar{t}=g_2(T_2\bar{t})$$

则存在唯一的态射$G:X_1\to X_2$使得

$$\bar{g_1}=\bar{g_2}G$$

$$T_2\bar{t}=\bar{\tilde{q}}G$$

要证(T_1,G,id_A)为正合列的同态，且有$\bar{g_1}=\bar{g_2}G$，则只需证$Gh_1=h_2T_1$即可。而

$$\bar{\tilde{q}}Gh_1=T_2\bar{t}h_1=T_2f_1=f_2T_1=\bar{\tilde{q}}h_2T_1$$

且由$\tilde{q}$为单态射可知$\bar{\tilde{q}}$，则有$Gh_1=h_2T_1$，从而(T_1,G,id_A)为正合列的同态。从T_1,id_A为同构，可得(T_1,G,id_A)为同构。 ■

对偶地，有

引理 7 *Abel*范畴$\mathcal{C}$中，有同构：

$$\begin{tikzcd}
0 \arrow[r] & (B\bigoplus B)\bigoplus B \arrow[r] \arrow[d, "s_1'"'] & C_1 \arrow[r] \arrow[d, "T_1"] & A_1 \arrow[r] \arrow[d, "T_2"] & 0 \\
0 \arrow[r] & B\bigoplus B\bigoplus B \arrow[r] & C_2 \arrow[r] & A_2 \arrow[r] & 0
\end{tikzcd}$$

其中$s_1' : (B \bigoplus B) \bigoplus B \to B \bigoplus B \bigoplus B$为自然同构。对第一列用$p_b(p_b \bigoplus id_B) : (B \bigoplus B) \bigoplus B \to B$拉回得到正合列：$0 \longrightarrow B \longrightarrow Y_1 \longrightarrow A_1 \longrightarrow 0$；第二列用$\widetilde{p}_b : B \bigoplus B \bigoplus B \to B$，则存在态射$H : Y_1 \to Y_2$使得有同构：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & Y_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow{\scriptstyle H} & & \Big\downarrow{\scriptstyle T_1} & & \\ 0 & \longrightarrow & B & \longrightarrow & Y_2 & \longrightarrow & A_2 & \longrightarrow & 0 \end{array}$$

对$A \bigoplus (A \bigoplus A)$也有相应结论

引理 8 *Abel*范畴$\mathcal{C}$中，有同构：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & A \bigoplus (A \bigoplus A) & \longrightarrow & 0 \\ & & {\scriptstyle T_1}\Big\downarrow & & \Big\downarrow{\scriptstyle T_2} & & \Big\downarrow{\scriptstyle s_2} & & \\ 0 & \longrightarrow & B & \longrightarrow & C_2 & \longrightarrow & A \bigoplus A \bigoplus A & \longrightarrow & 0 \end{array}$$

其中$s_2 : A \bigoplus (A \bigoplus A) \to A \bigoplus A \bigoplus A$为自然同构。对第一列用$(id_A \bigoplus q_a)q_a : A \to (A \bigoplus A) \bigoplus A$拉回得到正合列：$0 \longrightarrow B_1 \longrightarrow X_1 \longrightarrow A \longrightarrow 0$，第二列用$\widetilde{q}_a : A \to A \bigoplus A \bigoplus A$，则存在态射$G : X_1 \to X_2$使得有同构：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_1 & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & {\scriptstyle T_1}\Big\downarrow & & \Big\downarrow{\scriptstyle G} & & \Big\| & & \\ 0 & \longrightarrow & B_2 & \longrightarrow & X_2 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

引理 9 *Abel*范畴$\mathcal{C}$中，有同构：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B \bigoplus (B \bigoplus B) & \longrightarrow & C_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & {\scriptstyle s_2'}\Big\downarrow & & \Big\downarrow{\scriptstyle T_1} & & \Big\downarrow{\scriptstyle T_2} & & \\ 0 & \longrightarrow & B \bigoplus B \bigoplus B & \longrightarrow & C_2 & \longrightarrow & A_2 & \longrightarrow & 0 \end{array}$$

其中$s_2' : B \bigoplus (B \bigoplus B) \to B \bigoplus B \bigoplus B$为自然同构。对第一列用$p_b(id_B \bigoplus p_b) : B \bigoplus (B \bigoplus B) \to B$拉回得到正合列：$0 \longrightarrow B \longrightarrow Y_1 \longrightarrow A_1 \longrightarrow 0$，第二列用$\widetilde{p}_b : B \bigoplus B \bigoplus B \to B$，则存在态射$H : Y_1 \to Y_2$使得有同构：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & Y_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow{\scriptstyle H} & & \Big\downarrow{\scriptstyle T_2} & & \\ 0 & \longrightarrow & B & \longrightarrow & Y_2 & \longrightarrow & A_2 & \longrightarrow & 0 \end{array}$$

引理 10 *Abel*范畴$\mathcal{C}$中，$\xi_i : 0 \longrightarrow B \longrightarrow C_i \longrightarrow A \longrightarrow 0 (i = 1, 2, 3)$正合，$(\xi_1 * \xi_2) * \xi_3$得到正合列：$0 \longrightarrow B \longrightarrow Y_2 \longrightarrow A \longrightarrow 0$，而正合列：$0 \longrightarrow B \bigoplus B \bigoplus B \longrightarrow C_1 \bigoplus C_2 \bigoplus C_3 \longrightarrow A \bigoplus A \bigoplus A \longrightarrow 0$先用$\widetilde{q}_a : A \to A \bigoplus A \bigoplus A$拉回；再用$\widetilde{p}_b : B \bigoplus B \bigoplus B \to B$得到正合列：$0 \longrightarrow B \longrightarrow \overline{Y} \longrightarrow A \longrightarrow 0$，则$H : Y_2 \to \overline{Y}$ 使得有同构：

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & Y_2 & \longrightarrow & A & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow{\scriptstyle H} & & \Big\| & & \\ 0 & \longrightarrow & B & \longrightarrow & \overline{Y} & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

证明 对$i=1,2,3$

$$\xi_i: 0 \longrightarrow B \xrightarrow{f_i} C_i \xrightarrow{g_i} A \longrightarrow 0$$

$\xi_1 * \xi_2$：

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\overline{h_1}"] & Y_1 \arrow[r, "h_2"] & A \arrow[r] & 0 \\
0 \arrow[r] & B\bigoplus B \arrow[r, "h_1"] \arrow[u, "p_b"] \arrow[d, equal] & X_1 \arrow[r, "\overline{g_1\bigoplus g_2}"] \arrow[u, "\overline{p_b}"] \arrow[d, "\overline{q_a}"] & A \arrow[r] \arrow[u, equal] \arrow[d, "q_a"] & 0 \\
0 \arrow[r] & B\bigoplus B \arrow[r, "f_1\bigoplus f_2"] & C_1\bigoplus C_2 \arrow[r, "g_1\bigoplus g_2"] & A\bigoplus A \arrow[r] & 0
\end{tikzcd}$$

其中$(X_1,\overline{g_1\bigoplus g_2},\overline{q_a})$为拉回，$(Y_1,\overline{h_1},\overline{p_b})$为推出。
$(\xi_1*\xi_2)*\xi_3$：

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\overline{h_3}"] & Y_2 \arrow[r, "h_4"] & A \arrow[r] & 0 \\
0 \arrow[r] & B\bigoplus B \arrow[r, "h_3"] \arrow[u, "p_b"] \arrow[d, equal] & X_2 \arrow[r, "\overline{h_2\bigoplus g_3}"] \arrow[u, "\overline{p_b'}"] \arrow[d, "\overline{q_a'}"] & A \arrow[r] \arrow[u, equal] \arrow[d, "q_a"] & 0 \\
0 \arrow[r] & B\bigoplus B \arrow[r, "\overline{h_1}\bigoplus f_3"] & Y_1\bigoplus C_3 \arrow[r, "h_2\bigoplus g_3"] & A\bigoplus A \arrow[r] & 0 \\
0 \arrow[r] & (B\bigoplus B)\bigoplus B \arrow[r, "h_1\bigoplus f_3"] \arrow[u, "p_b\bigoplus id_B"] \arrow[d, equal] & X_1\bigoplus C_3 \arrow[r, "\overline{g_1\bigoplus g_2}\bigoplus g_3"] \arrow[u, "\overline{p_b}\bigoplus id_{C_3}"] \arrow[d, "\overline{q_a}\bigoplus id_{C_3}"] & A\bigoplus A \arrow[r] \arrow[u, equal] \arrow[d, "q_a\bigoplus id_A"] & 0 \\
0 \arrow[r] & (B\bigoplus B)\bigoplus B \arrow[r, "(f_1\bigoplus f_2)\bigoplus f_3"] & (C_1\bigoplus C_2)\bigoplus C_3 \arrow[r, "(g_1\bigoplus g_2)\bigoplus g_3"] & (A\bigoplus A)\bigoplus A \arrow[r] & 0
\end{tikzcd}$$

其中$(X_2,\overline{h_2\bigoplus g_3},\overline{q_a'})$为拉回，$(Y_1,\overline{h_3},\overline{p_b'})$为推出。

从命题3以及命题4可以得到方形$(X\bigoplus C_3,\overline{g_1\bigoplus g_2}\bigoplus g_3,\overline{q_a\bigoplus id_{C_3}})$为拉回，而方形$(Y_1\bigoplus C_3,\overline{h_1}\bigoplus f_3,\overline{p_b}\bigoplus id_B)$为推出。从而，$(\xi_1*\xi_2)*\xi_3$是对正合列$(\xi_1\bigoplus\xi_2)\bigoplus\xi_3$先用$q_a\bigoplus id_A$拉回，接着用$p_b\bigoplus id_B$推出，再用$q_a$拉回，最后用$p_b$推出得到。

由于q_a是单态射，$p_b\bigoplus id_B$是满态射，则由引理3，可知，先用$p_b\bigoplus id_B$推出，再用q_a拉回，可以换为先用q_a拉回，再用$p_b\bigoplus id_B$ 推出，在等价关系下得到的结果不变。

另外，从引理4可知，先用$q_a\bigoplus id_A$拉回，再用q_a拉回，可以直接用$(q_a\bigoplus id_A)q_a$拉回代替；从引理5可知，先用$p_b\bigoplus id_B$推出，再用p_b推出，可以直接用$p_b(p_b\bigoplus id_B)$推出代替。从而$(\xi_1*\xi_2)*\xi_3$：

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\overline{h_3}"] & Y_2 \arrow[r, "h_4"] & A \arrow[r] & 0 \\
0 \arrow[r] & (B\bigoplus B)\bigoplus B \arrow[r] \arrow[u, "p_b(p_b\bigoplus id_B)"] \arrow[d, equal] & X_2' \arrow[r, "w"] \arrow[u, "v"] \arrow[d, "r"] & A \arrow[r] \arrow[u, equal] \arrow[d, "(q_a\bigoplus id_A)q_a"] & 0 \\
0 \arrow[r] & (B\bigoplus B)\bigoplus B \arrow[r, "(f_1\bigoplus f_2)\bigoplus f_3"] & (C_1\bigoplus C_2)\bigoplus C_3 \arrow[r, "(g_1\bigoplus g_2)\bigoplus g_3"] & (A\bigoplus A)\bigoplus A \arrow[r] & 0
\end{tikzcd}$$

其中(X_2', w, r)为拉回，$(Y_2, \overline{h_3}, v)$为推出。再对$\xi_1 \bigoplus \xi_2 \bigoplus \xi_3$用$\widetilde{q_a}$拉回得到：

$$
\begin{tikzcd}
0 \arrow[r] & B \bigoplus B \bigoplus B \arrow[r] \arrow[d, equal] & \overline{X} \arrow[r, "g"] \arrow[d, "\overline{\widetilde{q_a}}"] & A \arrow[r] \arrow[d, "\widetilde{q_a}"] & 0 \\
0 \arrow[r] & B \bigoplus B \bigoplus B \arrow[r, "f_1 \bigoplus f_2 \bigoplus f_3"] & C_1 \bigoplus C_2 \bigoplus C_3 \arrow[r, "g_1 \bigoplus g_2 \bigoplus g_3"] & A \bigoplus A \bigoplus A \arrow[r] & 0
\end{tikzcd}
$$

其中$(\overline{X}, g, \overline{\widetilde{q_a}})$为拉回。

由推论3，有正合列的同构：

$$
\begin{tikzcd}
0 \arrow[r] & (B \bigoplus B) \bigoplus B \arrow[r, "(f_1 \bigoplus f_2) \bigoplus f_3"] \arrow[d, "s_1'"] & (C_1 \bigoplus C_2) \bigoplus C_3 \arrow[r, "(g_1 \bigoplus g_2) \bigoplus g_3"] \arrow[d, "s_1''"] & (A \bigoplus A) \bigoplus A \arrow[r] \arrow[d, "s_1"] & 0 \\
0 \arrow[r] & B \bigoplus B \bigoplus B \arrow[r, "f_1 \bigoplus f_2 \bigoplus f_3"] & C_1 \bigoplus C_2 \bigoplus C_3 \arrow[r, "g_1 \bigoplus g_2 \bigoplus g_3"] & A \bigoplus A \bigoplus A \arrow[r] & 0
\end{tikzcd}
$$

则根据引理6可知存在态射$G : X_2' \rightarrow \overline{X}$使得有下面正合列的同构：

$$
\begin{tikzcd}
0 \arrow[r] & (B \bigoplus B) \bigoplus B \arrow[r] \arrow[d, "s'"'] & X_2' \arrow[r] \arrow[d, "G"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \bigoplus B \bigoplus B \arrow[r] & \overline{X} \arrow[r] & A \arrow[r] & 0
\end{tikzcd}
$$

接着用$\widetilde{p_b}$推出：

$$
\begin{tikzcd}
0 \arrow[r] & B \bigoplus B \bigoplus B \arrow[r] \arrow[d, "\widetilde{p_b}"'] & \overline{X} \arrow[r] \arrow[d, "\overline{\widetilde{p_b}}"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "h"] & \overline{Y} \arrow[r] & A \arrow[r] & 0
\end{tikzcd}
$$

其中$(\overline{Y}, \overline{\widetilde{p_b}}, h)$为推出。则根据引理7可知存在唯一的态射$H : Y_2 \rightarrow \overline{Y}$使得有下面正合列的同构：

$$
\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & Y_2 \arrow[r] \arrow[d, "H"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & \overline{Y} \arrow[r] & A \arrow[r] & 0
\end{tikzcd}
$$

则引理成立。 ■

同样得对$\xi_1 * (\xi_2 * \xi_3)$也有

引理 11 *Abel*范畴$\mathcal{C}$中，$\xi_i : 0 \longrightarrow B \longrightarrow C_i \longrightarrow A \longrightarrow 0 (i = 1, 2, 3)$正合，$\xi_1 * (\xi_2 * \xi_3)$得到正合列：$0 \longrightarrow B \longrightarrow \overset{*}{Y}_2 \longrightarrow A \longrightarrow 0$，而正合列：$0 \longrightarrow B \bigoplus B \bigoplus B \longrightarrow C_1 \bigoplus C_2 \bigoplus C_3 \longrightarrow A \bigoplus A \bigoplus A \longrightarrow 0$先用$\widetilde{q_a} : A \rightarrow A \bigoplus A \bigoplus A$拉回；再用$\widetilde{p_b} : B \bigoplus B \bigoplus B \rightarrow B$得到正合列：$0 \longrightarrow B \longrightarrow \overline{Y} \longrightarrow A \longrightarrow 0$，则$F : \overset{*}{Y}_2 \rightarrow \overline{Y}$使得有同构：

$$
\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \overset{*}{Y}_2 \arrow[r] \arrow[d, "F"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & \overline{Y} \arrow[r] & A \arrow[r] & 0
\end{tikzcd}
$$

由引理10,11可得

定理 5 （结合律）*Abel*范畴$\mathcal{C}$中，$[\xi_1],[\xi_2]),[\xi_3] \in E(A,B)$，则$([\xi_1]+[\xi_2])+[\xi_3] = [\xi_1]+([\xi_2]+[\xi_3])$

至此从上面命题可以完成对E(A,B)为加法群的证明。

结论

文章给出了Ext^1的一般形式，与此类似的，也可以给出Ext^n的一般构造，从而得到Hom函子的导函子，继而得到长正合列。

致谢语

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